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# On the oblique interaction of a large and a small solitary wave

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The steady oblique interaction of two solitary waves on the surface of water of constant depth is considered. One wave is taken to be of arbitrary (large) amplitude and the other is small with a (non-dimensional) amplitude measured by the parameter $\epsilon$. A solution is sought as an asymptotic expansion, based on $\epsilon$, that assumes that in some region the solution is the sum of the two waves plus interaction terms. It is shown that this expansion is not uniformly valid close to a critical angle. This angle varies from zero (parallel waves) up to about 63·5°, as the amplitude of the larger wave increases from infinitesimally small to the largest-possible solitary wave. In the limit of two small waves, the details agree precisely with the results obtained by Miles (1977*a*).

When the angle between the two waves is close to the critical angle, for a given large wave, an alternative asymptotic expansion is required. In this strong-interaction case, the dominant term is just the large wave but with a phase shift that is an arbitrary function of the characteristic variable associated with the small wave. This function is determined by matching to an appropriate far field, and it turns out to be proportional to the logarithm of a hypergeometric function (which itself can be expressed in terms of the associated Legendre function $P_\nu^{-\mu}$). The phase shift is then well-defined (finite, real) provided the angle is not *very* close to the critical value. When this occurs the phase shift can be infinite at a specific angle (which corresponds to the case $|\mu| = \nu > 0$), and even closer to the critical angle ($|\mu| < \nu$) the phase shift is undefined (no longer real). A real solution for the wave profile is still possible if negative amplitudes are allowed, but the resulting solution is unacceptable since the surface is then not undisturbed at infinity. It is shown that the criterion $|\mu| < \nu$ matches exactly (for two small waves) with Miles' criterion for the non-existence of a regular reflection.

For the strong interaction (and $|\mu| \geqslant \nu$) it is argued that the small wave cannot penetrate the large wave unchanged. The large wave suffers significant distortion (bending) in the interaction, but the small wave, if it penetrates at all, must have an amplitude $o(\epsilon)$. This is the main aspect of the problem which cannot be completely determined using the present methods. The difficulty can be traced to the large solitary wave which is not known in closed form: only the exponential behaviour in the 'tail' is used explicitly in this work.

## 1. Introduction

The nonlinear interaction of waves has been, for some considerable time, a fascinating and rewarding field of study. Probably the most outstanding approach to emerge over the last decade is the inverse-scattering transform. This has enabled exact descriptions of various nonlinear wave phenomena to be obtained as solutions to certain rather special evolution equations. In the context of water waves the Korteweg–de Vries (KdV) equation holds a central position, the more so because the equation appears to predict readily identifiable types of interaction (for plane parallel waves). More recently the two-dimensional analogue of the KdV equation (2-D KdV) has facilitated the analysis of the oblique interaction of solitary waves (see e.g. Satsuma 1976).

In order to study the role of the 2-D KdV equation – and therefore the nonlinear interaction – in water-wave theory, Miles (1977*a*) considered the general interaction of two oblique solitary waves. He showed that an asymptotic expansion in amplitude was not uniformly valid when the two waves were nearly parallel; in this situation the waves and their interaction were of the same order of magnitude. Consequently a new solution was required for this 'strong' interaction, and it transpired that it was just the appropriate solution of the 2-D KdV equation. Miles also went on to point out that, in certain circumstances, a regular reflection (or interaction) was impossible. Now the results obtained by Miles, and in fact any discussion that uses a KdV equation at some stage, involves the approximation to small-amplitude long waves (see Johnson 1980). Unfortunately this restricts the application of the theories and perhaps overlooks some important (and new) phenomena. However, it must be said that these assumptions seem to be necessary if reasonably tractable problems are to be formulated.

Over the last year or two, the Department of Ocean Engineering at Newcastle has undertaken some work on the oblique interaction of waves that can arise in an inshore environment. For example, waves running in obliquely to a groyne or harbour wall suffer reflection, and it is then these reflected waves which interact with the incoming (incident) waves. The resulting wave profiles can be quite unusual in form, and can dramatically affect the onset of wave-breaking (Halliwell & Machen 1981). However, the observations and related experimental work are found to apply to régimes where the wave amplitude is in no sense small, at least for the incident wave. In fact the incident wave can be very close to breaking when the interaction occurs. The lack of available data on the interaction of larger-amplitude waves prompted the present study.

It is readily observed that, on occasions, the waves approaching a shoreline are fairly well-spaced one from another. This suggests that a first attempt at analysis should treat the waves as solitary. Of course there are situations when the waves must be regarded as both of large amplitude and periodic, but this will be beyond the scope of the work discussed here. If the incident wave is to be a large solitary wave, what of the reflected wave? Clearly, it would be most gratifying if this could also be of arbitrary (large) amplitude, but the resulting problem would appear to be amenable only to a numerical treatment. Mathematical expediency leads us immediately to consider a model in which the reflected wave is a small solitary wave. As it happens the reflected wave is rather smaller than the incident wave, owing to the loss of energy and momentum at the wall; in some circumstances it is found that the reflected wave can be quite small compared with the incident wave. It is also quite natural, if eventually we wish to study the interaction of two large waves, to first understand the problem of one large and one small wave. To this end we shall examine the oblique interaction of a large solitary wave with a small one, over a constant depth of water.

The large solitary wave cannot be expressed in closed form, although its shape and properties are well-known (see Longuet-Higgins & Fenton 1974; Byatt-Smith & Longuet-Higgins 1976; Cokelet 1977). Nevertheless, it is possible to discuss the perturbation of a large solitary wave due to the presence of a small (oblique) wave without specifying in detail the form of the larger wave. It is necessary merely to assume the existence of such a solution, and then use the amplitude of the smaller wave to define an appropriate expansion parameter. The general form of the interaction, and the non-uniformity in the expansion, can then be obtained by using only the exponential decay in the tail of the large wave. However, detailed properties of the interaction wave profile clearly require more information about the large wave itself. We shall show that the breakdown in the expansion is similar in character to that given in Miles (1977*a*), but that the resulting profile can be somewhat different.

## 2. Formulation and basic expansion

The fluid is assumed to be incompressible and irrotational, and bounded above by a free surface and below by a rigid horizontal surface. The fluid extends to infinity in all horizontal directions and the free surface is assumed to be a surface of constant pressure. In the absence of any disturbances the fluid is stationary with a constant depth $d$. Using only $d$ and the acceleration due to gravity $g$, the governing equations and boundary conditions are non-dimensionalized to yield

$$\frac{\partial^2\phi}{\partial z^2}+\nabla_\perp^2\phi = 0, \quad \text{with} \quad \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0; \tag{1a,b}$$

$$\left.\begin{aligned} \frac{\partial\phi}{\partial z} &= \frac{\partial\eta}{\partial t}+(\nabla_\perp\phi)\,.\,(\nabla_\perp\eta), && (2)\\ \eta+\frac{\partial\phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial z}\right)^2+(\nabla_\perp\phi)^2\right] &= 0 && (3)\end{aligned}\right\} \quad \text{on} \quad z = 1+\eta.$$

The vertical co-ordinate measured up from the bottom of the fluid is $z$, and the free surface is at $z = 1+\eta$. The gradient operator perpendicular to $z$ is denoted by $\nabla_\perp$, so that we may write

$$\nabla_\perp \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),$$

for example. Anticipating that the wave interaction will be a steady phenomenon (we exclude wave breaking), then the problem can be expressed in terms of two characteristic co-ordinates. Let the unit normals to the undistorted wave fronts be given by $\mathbf{n}_i$ $(i = 1,2)$ and introduce

$$\xi = \mathbf{n}_1.\mathbf{x}-F_1t, \quad \zeta = \epsilon^{\frac{1}{2}}(\mathbf{n}_2.\mathbf{x}-F_2t), \tag{4}$$

where $\mathbf{x} = (x,y)$, $F_i$ are Froude numbers† and $\epsilon$ is the (non-dimensional) amplitude of the smaller wave. The gradient operator now becomes

$$\nabla_\perp \equiv \mathbf{n}_1\frac{\partial}{\partial\xi}+\epsilon^{\frac{1}{2}}\mathbf{n}_2\frac{\partial}{\partial\zeta}; \tag{5}$$

† That is, non-dimensional propagation speeds.

whence (1)–(3) can be written as

$$\phi_{zz}+\phi_{\xi\xi}+2\epsilon^{\frac{1}{2}}\lambda\phi_{\xi\zeta}+\epsilon\phi_{\zeta\zeta}=0, \quad \text{with} \quad \phi_z=0 \quad \text{on} \quad z=0, \tag{6a,b}$$

$$\left.\begin{aligned}\phi_z &= -(F_1\eta_\xi+\epsilon^{\frac{1}{2}}F_2\eta_\zeta)+\phi_\xi\eta_\xi+\epsilon^{\frac{1}{2}}\lambda(\phi_\zeta\eta_\xi+\phi_\xi\eta_\zeta)+\epsilon\phi_\zeta\eta_\zeta, \quad (7)\\ \eta &- (F_1\phi_\xi+\epsilon^{\frac{1}{2}}F_2\phi_\zeta)+\tfrac{1}{2}[\phi_z^2+\phi_\xi^2+2\epsilon^{\frac{1}{2}}\lambda\phi_\xi\phi_\zeta+\epsilon\phi_\zeta^2]=0 \quad (8)\end{aligned}\right\} \text{ on } z=1+\eta,$$

where $\lambda = \mathbf{n}_1.\mathbf{n}_2 = \cos(\theta_1-\theta_2)$, and subscripts $\xi$, $\zeta$ and $z$ denote partial derivatives. Here $\theta_1-\theta_2$ measures the angle between the normals (and therefore between the wave fronts).

The large solitary wave is assumed to be the appropriate solution of the set

$$\phi_{zz}+\phi_{\xi\xi}=0, \quad \text{with} \quad \phi_z=0 \quad \text{on} \quad z=0, \tag{9a,b}$$

$$\phi_z=(\phi_\xi-F_1)\eta_\xi, \quad \eta-F_1\phi_\xi+\tfrac{1}{2}(\phi_z^2+\phi_\xi^2)=0 \quad \text{on} \quad z=1+\eta. \tag{10a,b}$$

The small solitary wave is then the corresponding solution of (9) and (10) when expressed according to the transformation

$$\xi\to\zeta\epsilon^{-\frac{1}{2}}; \quad \phi\to\epsilon^{\frac{1}{2}}\phi; \quad \eta\to\epsilon\eta; \quad F_1\to F_2. \tag{11}$$

The Froude number for this flow therefore satisfies $F_2 \sim 1$ as $\epsilon\to 0$. We now suppose that there is a region of $(\xi,\zeta)$-space where the solution of (6)–(8) can be expressed as an asymptotic expansion in $\epsilon$, as $\epsilon\to 0$, with the dominant terms given by the solution of (9) and (10). We note that, for the small wave, the expansion for $\phi$ will require a term $O(\epsilon^{\frac{1}{2}})$ and so a corresponding term must appear in the expansion for $\eta$ even though the amplitude of the small wave is $O(\epsilon)$. Thus we write, for $\epsilon\to 0$,

$$\phi \sim \phi_{\mathrm{I}}(\xi,z)+\epsilon^{\frac{1}{2}}\phi_{\mathrm{II}}(\zeta,z;\epsilon)+\sum_{n=1}^{\infty}\epsilon^{\frac{1}{2}n}\hat{\phi}_n(\xi,\zeta,z), \tag{12}$$

$$\eta \sim \eta_{\mathrm{I}}(\xi)+\epsilon\eta_{\mathrm{II}}(\zeta;\epsilon)+\sum_{n=1}^{\infty}\epsilon^{\frac{1}{2}n}\hat{\eta}_n(\xi,\zeta), \tag{13}$$

where $\{\phi_{\mathrm{I}},\eta_{\mathrm{I}}\}$ is the solution of (9) and (10), and $\{\phi_{\mathrm{II}},\eta_{\mathrm{II}}\}$ is the solution of (9) and (10) under (11). It is convenient to regard $\phi_{\mathrm{II}}$ and $\eta_{\mathrm{II}}$ as exact solutions and therefore these functions depend explicitly on $\epsilon$; at a later stage we can use suitable asymptotic approximations when examining higher-order terms. The interaction of the two waves is embodied in the functions $\hat{\phi}_n$ and $\hat{\eta}_n$.

The procedure is now quite straightforward: the expansions (12) and (13) are used in (6)–(8), and the surface boundary conditions are evaluated on $z = 1+\eta_{\mathrm{I}}(\xi)$ by assuming the existence of suitable Taylor expansions. The leading-order terms (as $\epsilon\to 0$) constitute the exact solution $\{\phi_{\mathrm{I}},\eta_{\mathrm{I}}\}$, and thereafter like powers of $\epsilon$ are collected together. The problems for each $\{\hat{\phi}_n,\hat{\eta}_n\}$ are, in principle, elementary for they require the solution of Laplace's equation with boundary values given on a known surface. However, it is clear that the higher-order problems cannot be expressed in a simple form, but we shall mention one or two results that turn out to be significant. We omit the details (which are quite lengthy), and record that the leading-order interaction terms can be written as

$$\hat{\phi}_1=\hat{\gamma}(\zeta)\phi_{\mathrm{I}\xi}, \quad \hat{\eta}_1=\hat{\gamma}(\zeta)\eta_{\mathrm{I}\xi}, \tag{14}$$

where $\hat{\gamma}(\zeta)$ is arbitrary. To describe the next order we introduce $\theta(\xi, z)$, which satisfies Laplace's equation, with $\theta_z = 0$ on $z = 0$, and

$$\phi_{\mathrm{I}z}\theta' + (\phi_{\mathrm{I}\xi})' \int_0^{1+\eta_{\mathrm{I}}} \theta_\xi \, dz = [(1+\lambda\gamma)(\eta_{\mathrm{I}} - F_1\phi_{\mathrm{I}\xi}) + \tfrac{3}{2}\lambda\gamma\eta_{\mathrm{I}}^2 + (\gamma - \lambda - 2\lambda\gamma F_1)\eta_{\mathrm{I}}\phi_{\mathrm{I}\xi}]' \quad \text{on} \quad z = 1+\eta_{\mathrm{I}}, \qquad (15a)$$

where the prime denotes the total derivative with respect to $\xi$. Further, we define $\hbar(\xi)$ such that

$$(F_1 - \phi_{\mathrm{I}\xi})\hbar = (1+\lambda\gamma)\phi_{\mathrm{I}\xi} + \lambda\gamma\eta_{\mathrm{I}}\phi_{\mathrm{I}\xi} + (\lambda - \gamma + \lambda\gamma F_1)\eta_{\mathrm{I}} + \int_0^{1+\eta_{\mathrm{I}}} \theta_\xi \, dz \quad \text{on} \quad z = 1+\eta_{\mathrm{I}}, \qquad (15b)$$

and then

$$\hat{\phi}_2 = \tfrac{1}{2}\hat{\gamma}^2\phi_{\mathrm{I}\xi\xi} + (\lambda\gamma z\phi_{\mathrm{I}z} + \theta)\eta_{\mathrm{II}0}, \qquad (16a)$$

$$\hat{\eta}_2 = \tfrac{1}{2}\hat{\gamma}^2\eta_{\mathrm{I}\xi\xi} + \hbar\eta_{\mathrm{II}0}, \qquad (16b)$$

provided

$$\frac{d\hat{\gamma}}{d\zeta} = \gamma\eta_{\mathrm{II}0}(\zeta) \quad (\text{or} \quad \hat{\gamma} = \gamma\phi_{\mathrm{II}0}(\zeta)), \qquad (17)$$

where $\gamma$ is an arbitrary constant and $\eta_{\mathrm{II}0}$, $\phi_{\mathrm{II}0}$ are the dominant terms in the expansions for $\eta_{\mathrm{II}}(\zeta;\epsilon)$, $\phi_{\mathrm{II}}(\zeta, z;\epsilon)$ as $\epsilon \to 0$. (Although $\eta_{\mathrm{II}0}$ is known completely, it should be remembered that $\phi_{\mathrm{II}0}$ is defined only to within an arbitrary constant.)

The terms in $\{\hat{\phi}_n, \hat{\eta}_n\}$ that involve $\hat{\gamma}$ are seen to be just those arising from the Taylor expansions of

$$\phi_{\mathrm{I}}(\xi + \epsilon^{\frac{1}{2}}\hat{\gamma}, z) \quad \text{and} \quad \eta_{\mathrm{I}}(\xi + \epsilon^{\frac{1}{2}}\hat{\gamma}) \quad \text{with} \quad \hat{\gamma} = \gamma\phi_{\mathrm{II}0}(\zeta), \qquad (18)$$

and so describe the bending of the characteristic, whereas the rest of the solution (in (16) for example) describes a more complicated interaction term, which to this order is proportional to $\eta_{\mathrm{II}0}$. The general form of the solution corresponds to that obtained by Miles (1977*a*) for two small solitary waves. However, the asymptotic expansions derived here do not enable the value of $\gamma$ (see (17)) to be determined uniquely, and so precise comparison with Miles' result is not possible at this stage of the analysis. We shall now show that $\gamma$ can be obtained by matching the expansions (12) and (13) with appropriate far-field expansions valid in the tail/precursor of the large wave. Only when $\gamma$ is known can we examine the basic expansions for any non-uniformities that might arise as $\lambda$ varies, i.e. as the angle between the waves is altered.

## 3. Far-field expansion

Although it is quite in order to examine the basic expansions as $|\xi| \to \infty$ directly, and thereby impose suitable boundedness conditions to determine $\gamma$, it turns out to be expedient to examine the far field explicitly. The reason for this is twofold. Firstly, a discussion of the near field involves the asymptotic behaviour of $\theta$ and $\hbar$ (see (15)) as $|\xi| \to \infty$, and it is rather more straightforward to obtain the equivalent information from the far field itself. Secondly – and this is the more telling argument – we shall eventually require these far-field expansions to permit matching to other far-field solutions valid only for special $\lambda$.

Before we can attempt the construction of the far field, we shall require the asymptotic form of the large solitary wave for $|\xi| \to \infty$. In fact this is well-known, and easy to derive, yielding the result that if $\eta_{\mathrm{I}} \sim a\,e^{\mp\alpha\xi}$ then

$$\phi_{\mathrm{I}} \sim \mp (aF_1/\sin\alpha)\, e^{\mp\alpha\xi}\cos\alpha z \quad \text{as} \quad \xi \to \pm\infty, \tag{19}$$

where

$$F_{\mathrm{I}} = (\tan\alpha)/\alpha \quad (0 < \alpha < \tfrac{1}{2}\pi) \tag{20}$$

is Stokes' result and it is supposed that $a(F_1)$ is known. Now from the original equations (6)–(8), or from the basic expansions (12) and (13), it is clear that the appropriate far field is where $\xi = O(\epsilon^{-\frac{1}{2}})$ and so we introduce $\xi = X\epsilon^{-\frac{1}{2}}$. The expansions for $\eta$ and $\phi$ are now assumed to take the form

$$\left.\begin{aligned} \eta &\sim \epsilon\eta_{\mathrm{II}}(\zeta;\epsilon) + \left[\sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n}\hat{H}_n(X,\zeta)\right]\exp\left(-\alpha|X|\epsilon^{-\frac{1}{2}}\right), \\ \phi &\sim \epsilon^{\frac{1}{2}}\phi_{\mathrm{II}}(\zeta,z;\epsilon) + \left[\sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n}\hat{\Phi}_n(X,\zeta,z)\right]\exp\left(-\alpha|X|\epsilon^{-\frac{1}{2}}\right), \end{aligned}\right\} \tag{21}$$

where exponentially small terms ($\exp(-n\alpha|X|\epsilon^{-\frac{1}{2}})$, $n = 2, 3, \ldots$) have been omitted. As before, the procedure is systematic, and follows the outline given in §2 except that in this region the surface boundary conditions may be expanded about $z = 1$. The problems at each order enable $\{\hat{\Phi}_n, \hat{H}_n\}$ to be found in terms of certain functions which themselves are not determined until later in the sequence. The first three sets of functions are described below, although the details are not included.

Using Stokes' result (20), we find that

$$\hat{H}_0 = \hat{A}_0, \quad \hat{\Phi}_0 = \mp(\hat{A}_0 F_1/\sin\alpha)\cos\alpha z, \tag{22}$$

where $\hat{A}_0(X,\zeta)$ is an arbitrary function. The next order yields

$$\left.\begin{aligned} \hat{H}_1 &= \hat{A}_1 \mp F_1\cot\alpha(F_1\hat{A}_{0X} + \hat{A}_{0\zeta}), \\ \hat{\Phi}_1 &= \pm(\hat{A}_1F_1/\sin\alpha)\cos\alpha z - (F_1/\sin\alpha)(\hat{A}_{0X} + \lambda\hat{A}_{0\zeta})\,z\sin\alpha z, \end{aligned}\right\} \tag{23}$$

where $\hat{A}_1(X,\zeta)$ is arbitrary, but with

$$(1-\lambda_c F_1)\hat{A}_{0X} + (\lambda - \lambda_c)\hat{A}_{0\zeta} = 0 \quad (\lambda \neq \lambda_c), \tag{24}$$

where

$$\lambda_c = \frac{2F_1}{1+F_1^2+\alpha^2F_1^4}, \tag{25}$$

and we have made use of $F_2 \sim 1$.

If we note that it must be possible to choose $\hat{A}_0 = \hat{A}_0(X)$, that is when $\eta_{\mathrm{II}} \equiv 0$ and only the large wave exists, the required solution of (24) is just $\hat{A}_0 =$ constant. Finally, the equations relating $\hat{H}_2$ and $\hat{\Phi}_2$ require that $\hat{A}_1(X,\zeta)$ takes a particular form, namely

$$\hat{A}_1 = \pm\left(\frac{\frac{1}{2}\hat{A}_0\lambda_c\alpha}{F_1}\right)\frac{1+2\lambda F_1+\alpha^2F_1^4}{\lambda-\lambda_c}\phi_{\mathrm{II}0}(\zeta). \tag{26}$$

The surface profile, for example, therefore becomes

$$\eta \sim \epsilon\eta_{\mathrm{II}} + \hat{A}_0\left[1 \pm \epsilon^{\frac{1}{2}}\lambda_c\frac{\alpha(1+2\lambda F_1+\alpha^2F_1^4)}{2F_1(\lambda-\lambda_c)}\phi_{\mathrm{II}0}\cdots\right]\exp\left(\mp\alpha X\epsilon^{-\frac{1}{2}}\right), \tag{27}$$

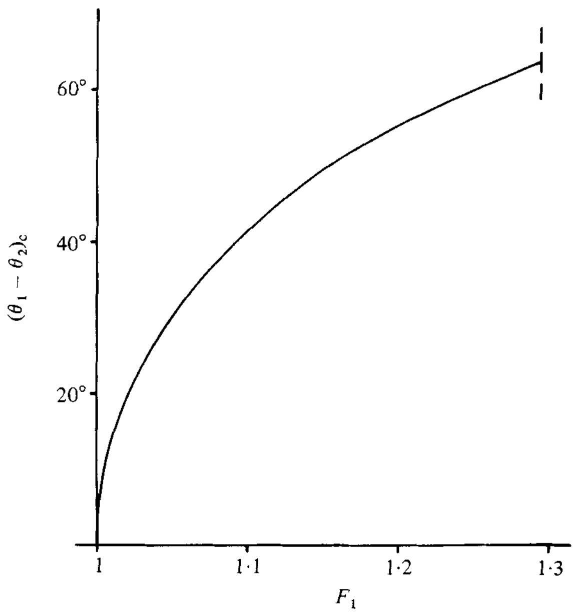


FIGURE 1. The variation of the critical angle with the Froude number $F_1$ ($\geqslant 1$) of the large wave.

and this is to match with

$$\eta \sim \epsilon\eta_{II} + \eta_I(\xi) + \epsilon^{\frac{1}{2}}\gamma\phi_{II0}\eta_{I\xi} \quad \text{as} \quad |\xi| \to \infty. \tag{28}$$

It is clear that these two expansions do indeed match if we choose

$$\text{(i)} \quad \hat{A}_0 = a \qquad \text{(see (19))}, \tag{29}$$

$$\text{(ii)} \quad \gamma = -\frac{\lambda_c(1+2\lambda F_1+\alpha^2 F_1^4)}{2F_1(\lambda-\lambda_c)}. \tag{30}$$

The comparison with Miles (1977*a*) can now be completed, and this serves as a check on the calculations presented so far. Since Miles considered two small waves, we must allow $\alpha \to 0$, $F_1 \to 1$ in order to describe the corresponding model here. From (25) it follows that $\lambda_c \to 1$, and then from (30) $\gamma \to (1+2\lambda)/2(1-\lambda)$ (which is just $\frac{3}{4}\kappa^{-1}-1$ when expressed in terms of $\kappa = \frac{1}{2}(1-\mathbf{n}_1.\mathbf{n}_2)$ as introduced by Miles). The agreement with Miles (1977*a*) is now complete in every detail (to this order), and in particular we can observe that both the basic and far-field expansions are not uniformly valid as $\lambda \to \lambda_c$: this is equivalent to the 'strong' interaction of Miles as $\lambda \to 1$. However, the angle at which the strong interaction occurs now depends on the amplitude of the large wave, and only for small-amplitude waves does the breakdown coincide with the waves becoming nearly parallel. Further, the large wave is limited by the wave of greatest height and so $\lambda_c = \cos(\theta_1-\theta_2)_c$ is in turn limited; the variation of the critical angle $(\theta_1-\theta_2)_c$, for values of $F_1$ up to about 1·29, is given in figure 1. As the amplitude increases so the critical angle varies from zero (parallel waves) to a maximum value of approximately 63·5°. If $\lambda$ is not close to $\lambda_c$ then the representations given in (12), (13) and (21) are uniformly valid and the solution describes two predominantly unaltered waves with a weak interaction, part of which slightly distorts the characteristic of the larger wave. However, if $\lambda - \lambda_c$ is small we must turn our attention to the construction of new basic and far-field expansions.

## 4. Strong interaction: basic expansion

The appearance of the non-uniformity in the far-field expansion, (27), or the basic expansion (28) with $\gamma$ given by (30), shows that these asymptotic expansions are not uniformly valid when $\lambda - \lambda_c = O(\epsilon^{\frac{1}{2}})$. Thus we introduce

$$\lambda = \lambda_c + \epsilon^{\frac{1}{2}}\Lambda, \tag{31}$$

where $\Lambda$ is a constant (positive or negative) which is assumed to be $O(1)$ as $\epsilon \to 0$. The form of (31) embodies some properties of both the large wave since $\lambda_c = \lambda_c(F_1)$ (noting that we have used $F_2 \sim 1$), and the small wave (via the amplitude parameter $\epsilon$). Although to leading order (31) is therefore dominated by the parameter $F_1$ of the large wave, we shall demonstrate that the character of the solution is crucially dependent on the precise value of $\Lambda$. Of course, for a pair of arbitrary-amplitude waves, we can anticipate that $\lambda_c = \lambda_c(F_1, \alpha_1; F_2, \alpha_2)$, symmetric in $F_1$, $F_2$ and $\alpha_1$, $\alpha_2$; (25) is the limiting form as $\alpha_2 \to 0$ ($F_2 \to 1$).

We may proceed with the basic (or near-field) expansion in much the same fashion as given in §2, but of course here $\lambda$ is expressed using (31). Also, it is clear that the solution in this region cannot be described in terms of the two basic waves

$$\eta_{\mathrm{I}}(\xi) + \epsilon\eta_{\mathrm{II}}(\zeta;\epsilon),$$

together with some suitable interaction. Consequently we seek a solution of the set (6)–(8), using (31), in the more general form

$$\eta \sim \sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n}\eta_n(\xi,\zeta), \quad \phi \sim \sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n}\phi_n(\xi,\zeta,z), \tag{32}$$

whence it is immediately evident that $\{\phi_0, \eta_0\}$ must satisfy the equations (9) and (10) defining the large solitary wave. Moreover, since these equations do not involve $\zeta$ explicitly, the most general solution available to us is

$$\eta_0(\xi,\zeta) \equiv \eta_{\mathrm{I}}(\xi + f(\zeta)), \quad \phi_0(\xi,\zeta,z) \equiv \phi_{\mathrm{I}}(\xi + f(\zeta), z), \tag{33}$$

where $f(\zeta)$ is arbitrary. Comparison with (18) indicates that (33) is of precisely the right form if matching is to be possible.

The next order is rather more involved. Since $\gamma$ appears in the complicated interaction term at $O(\epsilon)$ when $\lambda - \lambda_c = O(1)$ (see (16)), essentially the same type of interaction must now be evident at $O(\epsilon^{\frac{1}{2}})$ when $\lambda - \lambda_c = O(\epsilon^{\frac{1}{2}})$. Corresponding to (15)–(17) we introduce

$$\Theta_{zz} + \Theta_{\xi\xi} = 0, \quad \text{with} \quad \Theta_z = 0 \quad \text{on} \quad z = 0, \tag{34a}$$

$$\phi_{0z}\Theta' + (\phi_{0\xi})'\int_0^{1+\eta_0} \Theta_\xi\, dz = [\tfrac{1}{2}\lambda_c\eta_0^2 - (\lambda_c F_1 - 1)\,\eta_0\phi_{0\xi} - \lambda_c(1+\eta_0)\,(F_1\phi_{0\xi} - \eta_0)]' \quad \text{on} \quad z = 1 + \eta_0. \tag{34b}$$

If we define $H(\xi + f)$ such that

$$(F_1 - \phi_{0\xi})H = \lambda_c(1+\eta_0)\,\phi_{0\xi} + (\lambda_c F_1 - 1)\,\eta_0 + \int_0^{1+\eta_0} \Theta_\xi\, dz \quad \text{on} \quad z = 1 + \eta_0, \tag{35}$$

then

$$\eta_1 = f'(\xi)\,H(\xi+f), \quad \phi_1 = f'(\xi)\,[\lambda_c z\phi_{0z} + \Theta(\xi+f,z)]. \tag{36}$$

The solution for $\phi_1$ (i.e. $\Theta$) can always be shifted by an arbitrary function of $\zeta$, and so we can choose

$$\phi_1 \to \phi_{\mathrm{II}0}(\zeta) \quad \text{as} \quad \xi \to +\infty, \quad \text{say}. \tag{37}$$

That is since a wave interaction is being considered the small wave causing the distortion must exist somewhere: we choose it to be in the region $\xi > 0$. We make no assumption about the form of the small wave as $\xi \to -\infty$ (that is 'after' the interaction near $\xi = 0$) which requires the solution of (34) and (35). This is an important point to which we shall return later.

The other arbitrary function $f(\zeta)$ that appears here gives rise only to a bending of the characteristic of the large wave due to the presence of the smaller wave (see (33)), and therefore cannot contribute to the shape of the interaction profile (to leading order). The dominant term in this respect is $\epsilon^{\frac{1}{2}}f'H$, which again requires the solution of (34) and (35): it would seem to be impossible to extract any relevant details without recourse to a numerical treatment of (34) and (35). However, difficulties can also arise from $f(\zeta)$ for certain values of $\Lambda$; essentially this comes about because of the non-existence of a *real* $f(\zeta)$ for which matching can be accomplished. With this proviso (which we shall discuss in detail later), the matching of (33) to a suitable far-field solution should enable $f(\zeta)$ to be determined. Further, it should then also be possible to ascertain the nature of the profile in $\xi < 0$.

## 5. Strong interaction: far-field expansion

The far-field expansion is expressed, as in §3, by using $X = \epsilon^{\frac{1}{2}}\xi$ and then seeking a solution

$$\left.\begin{aligned} \eta &\sim \epsilon\eta_{\mathrm{II}}(\zeta;\epsilon) + \left[\sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n} H_n(X,\zeta)\right] \exp(-\alpha X\epsilon^{-\frac{1}{2}}), \\ \phi &\sim \epsilon^{\frac{1}{2}}\phi_{\mathrm{II}}(\zeta,z;\epsilon) + \left[\sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n} \Phi_n(X,\zeta,z)\right] \exp(-\alpha X\epsilon^{-\frac{1}{2}}), \end{aligned}\right\} \tag{38}$$

where only the dominant exponential terms are retained (see (21)), and matching is performed into the region $X > 0$ where the small wave $\{\phi_{\mathrm{II}}, \eta_{\mathrm{II}}\}$ is given. The leading-order problem then corresponds with (22), yielding

$$H_0 = A_0(X,\zeta), \quad \Phi_0 = -(A_0F_1/\sin\alpha)\cos\alpha z, \tag{39}$$

where $A_0(X,\zeta)$ is arbitrary. The next order requires that

$$A_{0X} = 0, \tag{40}$$

which follows when $\lambda_c$ (defined by (25)) is introduced. The solution for $\{\Phi_1, H_1\}$ can then be written as

$$\left.\begin{aligned} H_1 &= A_1 + F_1A_0'(\lambda_c\alpha F_1 - \cot\alpha), \\ \Phi_1 &= -(A_1F_1/\sin\alpha)\cos\alpha z - (\lambda_c F_1 A_0'/\sin\alpha)\,z\sin\alpha z, \end{aligned}\right\} \tag{41}$$

where $A_1(X,\zeta)$ is arbitrary and now $A_0 = A_0(\zeta)$, with the prime denoting a derivative. Finally, the equations that are satisfied by $\{\Phi_2, H_2\}$ imply that

$$A_{1X} = \hat{g}(\zeta), \tag{42}$$

where $\hat{g}(\zeta)$ is known in terms of $A_0(\zeta)$. Since (42) leads to a term $Xg(\zeta)$ in $A_1$, the expansions (38) are not uniformly valid as $X \to \infty$; we might surmise that it is necessary to choose $\hat{g}(\zeta) \equiv 0$. However a more careful examination is needed, before we accept this interpretation of (42), by constructing a 'far far field' where $\xi = O(\epsilon^{-1})$. This results in a partial differential equation for $A_0(\chi, \zeta)$, $\chi = \epsilon\xi$, which replaces (42):

$$C_3 A_{0\chi} = A_{0\zeta\zeta} - C_1 \Lambda A_{0\zeta} + C_2 \dot{\eta}_{\text{II}0} A_0. \tag{43}$$

The right-hand side of this equation is proportional to $\hat{g}(\zeta)$ in (42) and the $C_n$ $(n = 1, 2, 3)$ are constants (see below). As $|\zeta| \to \infty$ so the expansions (38) must match to the exponentially small regions of the large wave where the $\chi$-dependence is explicit, for example

$$\eta \sim a \exp(-\alpha\chi/\epsilon), \quad \text{as} \quad \zeta \to +\infty,$$

and this is possible only if $A_0(\chi, \zeta)$ is independent of $\chi$.† Hence the equation for $A_0(\zeta)$ (from (43) or (42)) is

$$A_0'' - C_1 \Lambda A_0' + C_2 \dot{\eta}_{\text{II}0} A_0 = 0, \tag{44}$$

where

$$\left.\begin{aligned} &C_1 = 2F_1\alpha/(\lambda_c c_4), \quad C_2 = \alpha^2(1 + 2F_1\lambda_c + \alpha^2 F_1^4)/C_4 \\ \text{and} \quad &C_3 = 2\alpha F_1(\lambda_c^{-1} - F_1)/C_4, \qquad (\text{see } (43)), \\ \text{with} \quad &C_4 = (1 - \lambda_c^2)(F_1/\lambda_c - 1) + 2\alpha^2 F_1^3 \lambda_c. \end{aligned}\right\} \tag{45}$$

The solution to (44) is to be determined such that it satisfies certain matching and boundedness conditions. These are as follows:

(i) as $\zeta \to +\infty$, $A_0 \to a$ to match;

(ii) as $\zeta \to -\infty$, $A_0$ is bounded to give no non-uniformity *between* the large and small waves ($\zeta < 0$, $\xi > 0$);

(iii) $|\Lambda| \to \infty$ to match with the far-field expansion valid for $\lambda - \lambda_c = O(1)$ (this serves as a check on (i) and (ii), and determines $\phi_{\text{II}0}$ uniquely);

(iv) $\xi \to +\infty$ to match to the asymptotic behaviour of the near field, which then determines $f(\zeta)$.

This final condition (iv) is the most significant of the matching conditions, for it enables the large wave to be completely determined. We shall describe these various conditions in a little more detail as we use them, but first it is convenient to re-cast the equation for $A_0(\zeta)$.

Equation (44) involves $\eta_{\text{II}0}(\zeta)$, which is the dominant representation of the small wave (which is given in $\xi > 0$), and this takes the form

$$\eta_{\text{II}0} = \operatorname{sech}^2(\tfrac{1}{2}\sqrt{3}\,\zeta). \tag{46}$$

The equation for $A_0(\zeta)$ therefore becomes

$$A_0'' - \sqrt{3}\,\mu A_0' + \tfrac{3}{4}\nu(1+\nu)\operatorname{sech}^2(\tfrac{1}{2}\sqrt{3}\,\zeta)\,A_0 = 0, \tag{47}$$

where we have written

$$C_1 \Lambda = \sqrt{3}\,\mu, \quad \nu^2 + \nu = \tfrac{4}{3}C_2. \tag{48$a, b$}$$

The parameter $\mu$ varies as the angle between the waves (measured at infinity) is altered. As the angle between the waves increases from zero (parallel) so the value of

† In fact an even more persuasive argument is used later (see (62)) which can be exploited here: $A_0(\chi, \zeta)$ diverges for either $\zeta \to +\infty$ or $-\infty$.

$\mu$ – which in this region is positive – decreases. At the critical angle (given by $\lambda = \lambda_c$) $\mu = 0$, and if the angle increases still further then $\mu < 0$ and increases in magnitude. The values of $\nu$ are fixed for a given large wave.

Now the appropriate solution of (47) can be obtained by first introducing

$$z = \tfrac{1}{2}(1 - \tanh \tfrac{1}{2}\sqrt{3}\,\zeta), \tag{49}$$

whence we have

$$z(1-z)\frac{d^2A_0}{dz^2} + (1+\mu-2z)\frac{dA_0}{dz} + \nu(1+\nu)A_0 = 0, \tag{50}$$

which is a hypergeometric equation. Let us first suppose that $\mu > \nu > 0$ (we may always choose $\nu > 0$, since our solution is independent of the specific root of (48*b*)); the solution that is bounded at both $z = 0, 1$ is

$$A_0 = K(a,\mu,\nu)\,F(1+\nu, -\nu; 1+\mu; z), \tag{51a}$$

where $K$ is an arbitrary constant. Now as $\zeta \to +\infty$ (i.e. $z \to 0$, for $\mu$, $\nu$ fixed) we enter the region ahead of the large wave ($\xi \to +\infty$) where (19) is valid, and so we require the matching condition (i). Thus we must choose

$$K = a. \tag{52}$$

If we apply the matching condition in $\mu$, i.e. $\Lambda$ (iii), then

$$A_0 \sim a\left(1 - \tfrac{4}{3}C_2\frac{z}{\mu}\right) \quad \text{as} \quad \mu \to +\infty;$$

so that from (38)

$$\eta \sim \epsilon\eta_{11} + a\left[1 - \frac{2C_2}{3\mu}(1 - \tanh\tfrac{1}{2}\sqrt{3}\,\zeta)\right]\exp(-\alpha X \epsilon^{\frac{1}{2}}),$$

which can be cast into a more recognizable form by observing that we may write

$$\phi_{\text{II}0} = -\int_{\zeta}^{\infty} \eta_{\text{II}0}(\zeta')\,d\zeta' = -2\sqrt{\tfrac{1}{3}}(1 - \tanh\tfrac{1}{2}\sqrt{3}\,\zeta),$$

which fixes the arbitrary constant in $\phi_{\text{II}0}$. The expansion (38) therefore becomes

$$\eta \sim \epsilon\eta_{\text{II}} + a\left(1 + \frac{C_2}{C_1\Lambda}\phi_{\text{II}0}\right)\exp(-\alpha X\epsilon^{\frac{1}{2}}) \quad \text{as} \quad \Lambda \to \infty,$$

and this matches precisely with (27) (upper sign) when we use (45) and (29). Thus the solution (51*a*), with (52), matches to the adjacent far field where $\xi > 0$ and $\lambda - \lambda_c = O(1)$. (The cases that arise when $0 < \mu \leqslant \nu$ will be dealt with in §6, but we can note that the argument just outlined is still applicable even though $A_0$ may not be positive for all $\zeta$.)

The situation for $\mu < 0$, and in particular for $\mu < -\nu$ ($\nu > 0$), can be described most conveniently by introducing

$$z' = \tfrac{1}{2}(1 + \tanh\tfrac{1}{2}\sqrt{3}\,\zeta),$$

whence the solution for $A_0$ that is bounded at both $z' = 0, 1$ is just

$$A_0 = K'(a,\mu,\nu)\,F(1+\nu, -\nu; 1-\mu; z').$$

Now the matching condition (i) requires

$$K' = \frac{a\Gamma(1+\nu-\mu)\,\Gamma(-\nu-\mu)}{\Gamma(-\mu)\,\Gamma(1-\mu)} \quad (\mu < -\nu < 0),$$

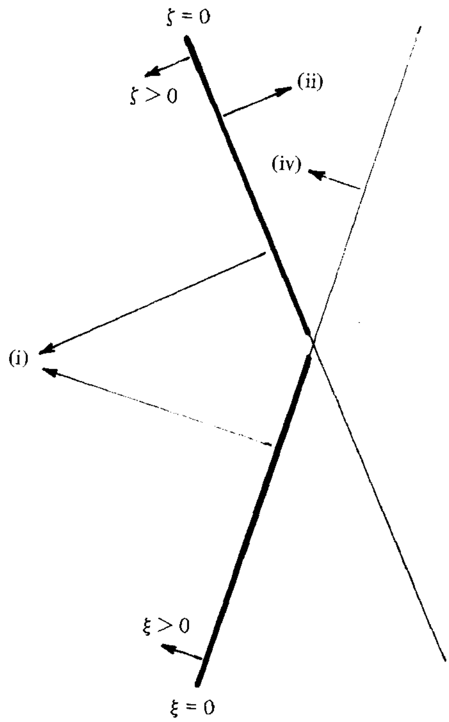


FIGURE 2. Diagrammatic representation of the wave fronts: large wave along $\xi = 0$, small wave along $\zeta = 0$. The given waves (in $\zeta > 0$, $\xi > 0$) are denoted by the heavier lines. The spatial matching and boundedness conditions are indicated by the arrows numbered (i), (ii), (iv); for details see text.

and then (iii) is satisfied if we choose

$$\phi_{\mathrm{II}0} = \int_{-\infty}^{\zeta} \eta_{\mathrm{II}0}(\zeta')\,d\zeta' = 2\sqrt{\tfrac{1}{3}}\,(1+\tanh\tfrac{1}{2}\sqrt{3}\,\zeta),$$

noting that $K' \to a$ as $\mu \to -\infty$. Since $-\infty < \zeta < \infty$ implies $1 > z > 0$ and $0 < z' < 1$ (both ordered with $\zeta$), the solution

$$A_0 = \frac{a\Gamma(1+\nu-\mu)\,\Gamma(-\nu-\mu)}{\Gamma(-\mu)\,\Gamma(1-\mu)}\,F(1+\nu, -\nu; 1-\mu; z') \tag{51b}$$

is the mirror image of (51$a$) in the line $\lambda = \lambda_c$ ($\mu = 0$), up to a multiplicative constant. Thus, at least for $|\mu| > \nu$, it is sufficient to discuss the solution (51$a$), which can be rewritten as

$$A_0 = a\Gamma(1+\mu)\left(\frac{1+T}{1-T}\right)^{\frac{1}{2}\mu} P_\nu^{-\mu}(T), \quad T = \tanh\tfrac{1}{2}\sqrt{3}\,\zeta, \tag{53}$$

where $P_\nu^{-\mu}$ is the associated Legendre function. This alternative to (51$a$) will prove useful in our later discussion.

Finally, the matching condition (iv) requires

$$\eta \sim \epsilon\eta_{\mathrm{II}} + A_0(\zeta)\exp(-\alpha X\epsilon^{\frac{1}{2}})$$

and

$$\eta \sim \epsilon\eta_{\mathrm{II}} + a\exp[-\alpha(\xi+f)]$$

to be identical. This is possible, provided $A_0 \geqslant 0$, if we choose

$$\alpha f(\zeta) = \ln(a/A_0(\zeta)). \tag{54}$$

The form – and in particular the location – of the large-wave solution is now completely determined, and it remains for us to discuss the consequences of using (54), and also the nature of the solution in $\xi < 0$. Both of these aspects turn out to be quite involved. However, so far, the analysis has been quite straightforward with the application of suitable matching conditions; for reference the spatial matching employed here is represented diagrammatically in figure 2.

Let us now consider the structure of the solution in $\xi < 0$. This proves to be a real difficulty, mainly because we do not know the solution to (34), (35). Consequently any statements must be, of necessity, rather incomplete: the solution of (34), (35) as $\xi \to -\infty$ would enable a comprehensive matching into the far field, $X < 0$. As it is, we must address this problem by examining the far field alone, but with the given matching condition involving $f(\zeta)$. Let us first suppose that the asymptotic expansion for $X < 0$ corresponds to (38), so that

$$\eta \sim \epsilon g(\zeta;\epsilon) + \left[\sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n} \tilde{H}_n(X,\zeta)\right] \exp(\alpha X \epsilon^{\frac{1}{2}}) \tag{55}$$

for example. It then follows that

$$g(\zeta;\epsilon) \equiv \eta_{\mathrm{II}}(\zeta+\zeta_0;\epsilon) \tag{56}$$

where $\zeta_0$ (a constant) is an unknown phase shift, and also $\tilde{H}_0(X,\zeta) = \tilde{A}_0(\zeta)$, where

$$\tilde{A}_0'' + C_1 \Lambda \tilde{A}_0' + C_2 \operatorname{sech}^2(\tfrac{1}{2}\sqrt{3}\,(\zeta+\zeta_0))\tilde{A}_0 = 0 \quad (\mu > 0 \text{ say}) \tag{57}$$

(see (47), (48)). The matching condition for $\tilde{A}_0(\zeta)$ is that

$$a \exp(\alpha f) = \tilde{A}_0(\zeta), \tag{58}$$

where $f(\zeta)$ is already known (see (54)), and so

$$A_0(\zeta)\tilde{A}_0(\zeta) = a^2 \quad \text{(const)}. \tag{59}$$

However, it is readily verified that (57) does not admit a solution $1/A_0$ for any $\zeta_0$ *unless* $\nu = 1$; we shall return to this special case later. Certainly, if $\nu \neq 1$, then an alternative asymptotic expansion must be sought.

The simplest generalization is to replace (57) by a partial differential equation, for a wider class of solution is then available. In particular, if we construct the corresponding equation to (43), in the far far field $\xi = O(\epsilon^{-1})$, then the matching condition becomes an initial condition on $\chi = 0$. It turns out that $\tilde{A}_0(\chi,\zeta)$ now satisfies

$$-C_3 \tilde{A}_{0\chi} = \tilde{A}_{0\zeta\zeta} + C_1 \Lambda \tilde{A}_{0\zeta} + C_2 \eta_{\mathrm{II}0}(\zeta+\zeta_0)\tilde{A}_0, \tag{60}$$

where $\chi = \epsilon\xi$, the constants are as given in (45), and

$$\tilde{A}_0(0,\zeta) = a\exp(\alpha f) = a^2/A_0(\zeta). \tag{61}$$

The formal solution can be written

$$\tilde{A}_0(\chi,\zeta) = \int_0^{\infty} A(\sigma)\tilde{F}(\zeta;\sigma)\exp(\sigma\chi + \tau\zeta)\,d\sigma \quad (\chi < 0), \tag{62}$$

where $A(\sigma)$ is supposed chosen so that (61) is satisfied, and

$$\tilde{F}(\zeta;\sigma) = F(1+\nu, -\nu; 1+\sqrt{\tfrac{1}{3}}\,(2\tau+C_1\Lambda); z) \quad \text{with} \quad \tau^2+C_1\Lambda\tau+C_3\sigma = 0.$$

But $\mathscr{R}(\tau) \lesseqgtr 0$ as $\Lambda \gtrless 0$, and so the solution (62) diverges as $\zeta \to \mp\infty$ (vertically ordered) and hence does not exist; matching as $|\zeta| \to \infty$ is once again impossible.

We have therefore found that matching to the near field – although the details are not clear – certainly cannot be accomplished if $g = O(1)$ $(\nu \neq 1)$, $\epsilon \to 0$. Let us suppose that $g$ is larger than $O(1)$; then from (60) we see that $\tilde{A}_0 \equiv 0$ which again does not allow matching. Thus we must perforce assume that $g = o(1)$ in (55); the equation for $\tilde{A}_0(\chi, \zeta)$ becomes

$$-C_3\tilde{A}_{0\chi} = \tilde{A}_{0\zeta\zeta}+C_1\Lambda\tilde{A}_{0\zeta} \quad (\chi < 0), \tag{63}$$

with the initial condition (61). The solution is

$$\tilde{A}_0(\chi,\zeta) = \tfrac{1}{2}a\left(\frac{-C_3}{\pi\chi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left[\alpha f(y)+C_3(Y-y)^2/4\chi\right]dy, \tag{64}$$

where $Y = \zeta - C_1\Lambda\chi/C_3$, and (64) does permit matching as $|\zeta| \to \infty$, since

$$\tilde{A}_0(\chi,\zeta) \to a\exp\left[\alpha f(\zeta)\right].$$

Because this approach appears to be the only one consistent with matching to the near field we can expect that (55) will now match with (27) as $\Lambda \to \infty$. However, this is possible only if $g(\zeta;\epsilon)$ matches as $|\Lambda| \to \infty$, i.e. if $g \to \eta_{\mathrm{II}0}(\zeta)$ as $|\Lambda| \to \infty$, and thus $g$ cannot be uniformly $o(1)$. Unfortunately we are unable to determine $g$; this is essentially because we have no solution of (34), (35). To proceed further we shall *assume* that $g$ matches (and otherwise our solution is evidently satisfactory). We introduce $X = \chi\epsilon^{-\frac{1}{2}}$, then as $\epsilon \to 0$

$$\hat{A}_0 \sim a\exp\left[\alpha f(\zeta)\right] = a^2/A_0(\zeta) \sim a\left(1-\frac{C_2}{C_1\Lambda}\phi_{\mathrm{II}0}\right) \quad \text{as} \quad |\Lambda| \to \infty,$$

which does match with (27) (lower sign). Thus the only solution consistent with the equations and matching conditions (modulo the form of $g$) describes a small wave which essentially does not penetrate into $\xi < 0$. However, any vestigial small wave is still evident by its interaction with the exponential tail of the large wave. There is some observational evidence to suggest the disappearance of a wave front after' an interaction, although this type of solution should not be regarded as analogous to the resonant triad (as relevant in the 2-D KdV equation for example).

The case $\nu = 1$ is worthy of closer inspection. If $g$ satisfies (56) then (57) becomes

$$\tilde{A}_0''+\sqrt{3}\,\mu\tilde{A}_0'+\tfrac{3}{2}\operatorname{sech}^2\left(\tfrac{1}{2}\sqrt{3}\,(\zeta+\zeta_0)\right)\tilde{A}_0 = 0,$$

which has the solution

$$\tilde{A}_0 = \frac{a}{\mu-1}\left[\mu-\tanh\left(\tfrac{1}{2}\sqrt{3}\,(\zeta+\zeta_0)\right)\right] \quad (\mu > 1 \quad \text{if} \quad \tilde{A}_0 > 0),$$

where $\tilde{A}_0 \to a$ as $\zeta \to +\infty$. Also, from (53), it is clear that

$$A_0 = \frac{a}{\mu+1}\left[\mu+\tanh\tfrac{1}{2}\sqrt{3}\,\zeta\right] \quad (\mu > \nu = 1),$$

and then $A_0\tilde{A}_0 = a^2$ if

$$\tanh\tfrac{1}{2}\sqrt{3}\,\zeta_0 = 1/\mu \quad (\mu > 1).$$

So, when $\nu = 1$, there does exist a solution for which the small wave is present in $\xi < 0$, but such a solution requires $g = O(1)$ as $\epsilon \to 0$. Now we have seen already that we require $g = o(1)$ with $g \to \eta_{\text{II}0}$ as $|\Lambda| \to \infty$, and thus the solution just outlined is applicable if also $\nu \to 1$: exactly these conditions pertain as $\alpha \to 0$. From (45) and (48) it follows that $\nu \to 1$ and $C_1 \sim 1/\alpha$ as $\alpha \to 0$, i.e. $|\mu| \to \infty$ (which corresponds to the result of allowing $|\Lambda| \to \infty$, $\alpha$ fixed). Since the limit involves $\alpha \to 0$ we can anticipate a further match with the Miles solution. This is easily verified, for the large wave then becomes

$$\eta_{\text{I}} \sim \epsilon_1 \operatorname{sech}^2 \left[ \tfrac{1}{2}(3\epsilon_1)^{\frac{1}{2}} \xi - \frac{3}{2} \frac{(\epsilon\epsilon_1)^{\frac{1}{2}}}{1-\lambda} (1-T) \right], \quad \alpha = (3\epsilon_1)^{\frac{1}{2}} \to 0, \quad T = \tanh\left(\tfrac{1}{2}\sqrt{3}\,\zeta\right);$$

and using the phase shift of the smaller wave this yields

$$\eta_{\text{II}} \sim \operatorname{sech}^2 \left[ \tfrac{1}{2}\sqrt{3}\,\zeta - 3\frac{(\epsilon\epsilon_1)^{\frac{1}{2}}}{1-\lambda} \right] \quad \text{as} \quad \xi \to -\infty.$$

Both these results match the Miles solution, for example by taking the strong-interaction limit ($\kappa \to 0$) of the weak-interaction solution in Miles (1977*a*). (It turns out (§6) that $\nu = 1$ at $F_1 \simeq 1{\cdot}225$, but this does not involve the limit $\alpha \to 0$; thus $\Lambda$ (or $\mu$) remains $O(1)$ and so, under our assumption, $g = o(1)$ and the solution above is not relevant.)

With the small wave (in $\xi > 0$) close to the critical angle, the interaction with the large wave produces a change of $O(1)$ in the form of that wave. In particular, the wave front is given by $\xi + f(\zeta) = 0$, to leading order, which implies a phase shift of

$$[f(\zeta)]_{-\infty}^{+\infty} = -f(-\infty).$$

Since for our solution we require $g = o(1)$, one is tempted to envisage the small wave losing its identity in its Herculean effort to bend the large wave.

## 6. Strong interaction: the role of $f(\zeta)$ and its breakdown

The dominant behaviour resulting from the oblique interaction of a large and small solitary wave has now resolved itself into a discussion of the phase shift $f(\zeta)$. It has already been noted that $f(\zeta)$ can be defined by (54) only if $A_0 \geqslant 0$; the possibility that $A_0$ might change sign clearly becomes a central issue. To this end we must examine the relevant properties of $P_\nu^{-\mu}$, using appropriate values of $\mu$ and $\nu$. If $A_0$ can indeed change sign – it is certainly positive in a neighbourhood of $T = 1$, see (53) – then an alternative matching procedure might have to be adopted. If $A_0 > 0$ everywhere then $f(\zeta)$ can be determined from (53) and (54), and in particular the phase shift can be obtained: this is all quite straightforward. If, on the other hand, $A_0 = 0$ at $T = -1$ ($\zeta = -\infty$) then the phase shift is essentially infinite and the interaction profile now does correspond to the resonant case of Miles (1977*b*). Some typical results for both $A_0 > 0$ ($-1 \leqslant T \leqslant 1$) and $A_0 = 0$ (at $T = -1$) will be presented in §7.

The associated Legendre function, for arbitrary degree $\nu$ and order $\mu$, cannot be expressed in a simple closed form. However, the distribution of zeros is well-known (see Hobson 1931), as well as the asymptotic behaviours near $T = \pm 1$ (see also Erdélyi *et al.* 1953). For the application intended here we can note that $\nu$ is determined from (48*b*) for a given large wave, i.e. for a fixed $F_1$ (which also prescribes $\lambda_c$). The parameter $\mu$ is proportional to $\Lambda$, and therefore represents the angle between the two

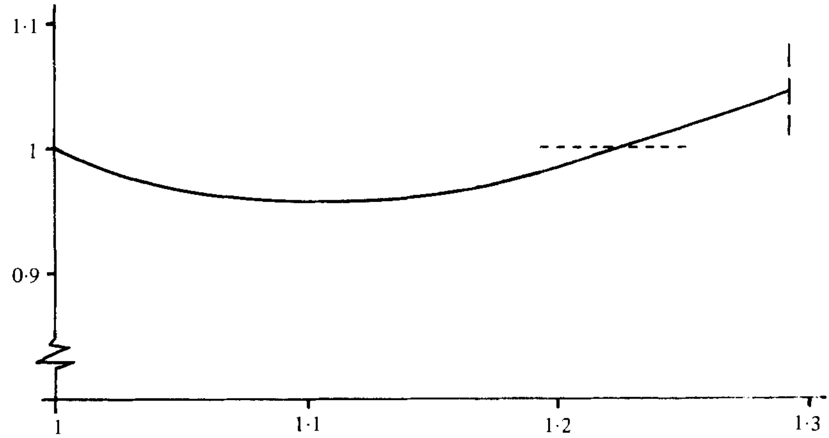


FIGURE 3. The variation of the degree $\nu$ of the associated Legendre function with the Froude number $F_1$ ($\geqslant 1$) of the large wave. (The curve crosses $\nu = 1$ (again) at $F_1 \simeq 1.225$.)

| | $0 < \nu \leqslant 1$ | $1 < \nu < 2$ | Number of zeros and positions |
|---|---|---|---|
| (a) | $\|\mu\| > \nu$ | $\|\mu\| > \nu$ | 0 |
| (b) | $\|\mu\| = \nu$ | $\|\mu\| = \nu$ | 1, at $T = -1$ |
| (c) | $0 \leqslant \|\mu\| < \nu$ | $\nu - 1 < \|\mu\| < \nu$ | 1, on $(-1, 1)$ |
| (d) | — | $\|\mu\| = \nu - 1$ | 2, at $T = -1$ and on $(-1, 1)$ |
| (e) | — | $0 \leqslant \|\mu\| < \nu - 1$ | 2, on $(-1, 1)$ |

TABLE 1. Number and positions of zeros of $A_0(T)$.

wave fronts (measured at $\xi$, $\zeta \to +\infty$). First, the value of $\nu$ is found from (48*b*) using (45), (25) and (20); the variation of $\nu$ with $F_1$ is shown in figure 3. (The curve crosses $\nu = 1$ at about $F_1 = 1.225$.) The highest wave corresponds to a value of about 1·04, and $\nu$ otherwise lies in the range $0.96 \leqslant \nu \leqslant 1.04$, approximately. The limit of infinitesimal waves occurs where $\nu \to 1$ as $F_1 \to 1$, which is therefore the neighbourhood applicable to the Miles (1977*a*) theory (see §5).

The number of zeros of $P_\nu^{-\mu}(T)$ on $[-1, 1]$ may be described for varying $\mu$ at fixed $\nu$, bearing in mind the allowed values of $\nu$. These results are used to give table 1 for the two cases of interest, namely $0 < \nu \leqslant 1$ and $1 < \nu < 2$, and by virtue of the symmetry in $\lambda = \lambda_c$ (see (51*a, b*)) we can combine both $\mu > 0$, $\mu < 0$. As the angle between the waves varies from nearly zero so $|\mu|$ decreases, but provided $|\mu| \geqslant \nu$ no zeros appear on $(-1, 1]$. As $|\mu|$ decreases still further, at most two zeros will occur. Thus for cases (*a*) and (*b*) the matching condition (54) is valid, that is whenever $|\mu| \geqslant \nu$. The remaining cases (*c*)–(*e*), involve a solution (for $A_0$) which changes sign at a finite value of $\zeta$; in these circumstances $f(\zeta)$ is no longer real for some $\zeta$, and hence the matching procedure must be reassessed.

First, however, let us consider the path of the large wave; in particular the peak of the wave lies along $\xi + f(\zeta) = 0$, to leading order. As $\zeta \to +\infty$, so $f(\zeta) \to 0$ and the peak is along $\xi = 0$ as originally given. Now let $A_0(\zeta)$ first be zero at $\zeta = \zeta_1$ (finite), i.e. $A_0(\zeta) > 0$ for $\zeta > \zeta_1$, whence $f(\zeta) \to +\infty$ as $\zeta \to \zeta_1$ and the peak of the wave asymptotes to $\zeta = \zeta_1$ as $\xi \to -\infty$ (see figure 4*d*). Along $\zeta = \zeta_1$ the wave is of zero amplitude and for $\zeta < \zeta_1$ the solution is no longer defined (in the sense that $f(\zeta)$ is no longer real). This leads us to examine the profile in the neighbourhood of $\zeta = \zeta_1$, and from the structure of the original equations – or the asymptotic expansions already discussed – we consider the region where the amplitude of the large wave is $O(\epsilon^{\frac{1}{2}})$ as $\xi \to +\infty$. Thus we transform under $\xi \to -(1/2\alpha)\ln\epsilon + \xi$, which therefore ensures that both $\eta$ and $\phi$ are $O(\epsilon^{\frac{1}{2}})$, and expand

$$\left.\begin{aligned} \eta &\sim \epsilon^{\frac{1}{2}}\left[\epsilon^{\frac{1}{2}}\eta_{\mathrm{II}}(\zeta;\epsilon) + \sum_{n=0}^{\infty}\epsilon^{\frac{1}{2}n}h_n(\xi,\zeta)\right], \\ \phi &\sim \epsilon^{\frac{1}{2}}\left[\phi_{\mathrm{II}}(\zeta;\epsilon) + \sum_{n=0}^{\infty}\epsilon^{\frac{1}{2}n}\Psi_n(\xi,\zeta,z)\right] \end{aligned}\right\} \tag{65}$$

as $\epsilon \to 0$. If these expansions are to match with (32), as $\alpha f \to +\infty$, then the exponential terms (in $\xi$) require that

$$h_n = \sum_{m=1}^{n+1} A_{nm}(\zeta)\,e^{-m\alpha\xi}, \quad \Psi_n = \sum_{m=1}^{n+1}\psi_{nm}(\zeta,z)\,e^{-m\alpha\xi}.$$

The equations involving $\{\psi_{01}, A_{01}\}$, $\{\psi_{11}, A_{11}\}$ and $\{\psi_{21}, A_{21}\}$ then follow from (39), (41) and (44) respectively, and in particular

$$A_{01}'' - C_1\Lambda A_{01}' + C_2\eta_{\mathrm{II}0}A_{01} = 0, \tag{66}$$

where $C_1$, $C_2$ are given in (45). The required solution to match (for $\mu \geqslant 0$, say) is just (53), so $A_{01}(\zeta) \equiv A_0(\zeta)$. In other words the matching condition is used to determine the amplitude of the wave profile directly, and since $A_0(\zeta)$ can be negative the surface profile can lie below the undisturbed level. Is this a physically realistic solution? In the region $\xi > 0$ (where the matching has been performed) the solution is at most $O(\epsilon^{\frac{1}{2}})$ and decays exponentially as $\xi \to +\infty$, at fixed $\zeta$. The amplitude remains bounded in $\zeta$ for $\mu \neq 0$, although if $\mu = 0$ the amplitude grows as $\zeta \to -\infty$: this special difficulty is not worthy of further consideration because of the behaviour that pertains in $\xi < 0$, as we shall see. The asymptotic solution (65) is valid for all $\xi$; this is quite easy to justify since the matching to the dominant exponential term $e^{-\alpha\xi}$ is valid even if $\xi < 0$. As $f \to +\infty$ $(\zeta \to \zeta_1)$ so

$$\eta_{\mathrm{I}} \sim a\exp[-\alpha(\xi + f)],$$

and $\xi$ is arbitrary (but finite). In consequence the matching just outlined to determine $A_{01}(\zeta)$, which requires terms $e^{-m\alpha\xi}$, is still applicable.† However, the solution in $\xi < 0$ with $\zeta < \zeta_1$ is wholly – or in part – negative and increasing as $\xi \to -\infty$; this cannot describe an undisturbed surface at infinity (and away from the wave fronts). Thus the solution for $|\mu| < \nu$ is not consistent with the given conditions at infinity. If $|\mu| < \nu$ then either no steady solution exists, or at least no steady solution of the form discussed here is possible. This situation is analogous to the non-existence of a regular reflection encountered by Miles (1977*a*). In fact it is straightforward to demonstrate that our criterion $|\mu| < \nu$ and Miles' match. From Miles (1977*a*) we have that there is no regular reflection if

$$(\alpha_2^{\frac{1}{2}} - \alpha_1^{\frac{1}{2}})^2 < \tfrac{4}{3}\sin^2\tfrac{1}{2}(\theta_1 - \theta_2) < (\alpha_2^{\frac{1}{2}} + \alpha_1^{\frac{1}{2}})^2,$$

† Alternatively we can examine the neighbourhood $\zeta - \zeta_1 = O(\epsilon^{\frac{1}{2}})$, where $\eta = O(\epsilon^{\frac{1}{2}})$, for arbitrary $\xi$; appropriate matching leads to the same conclusion.

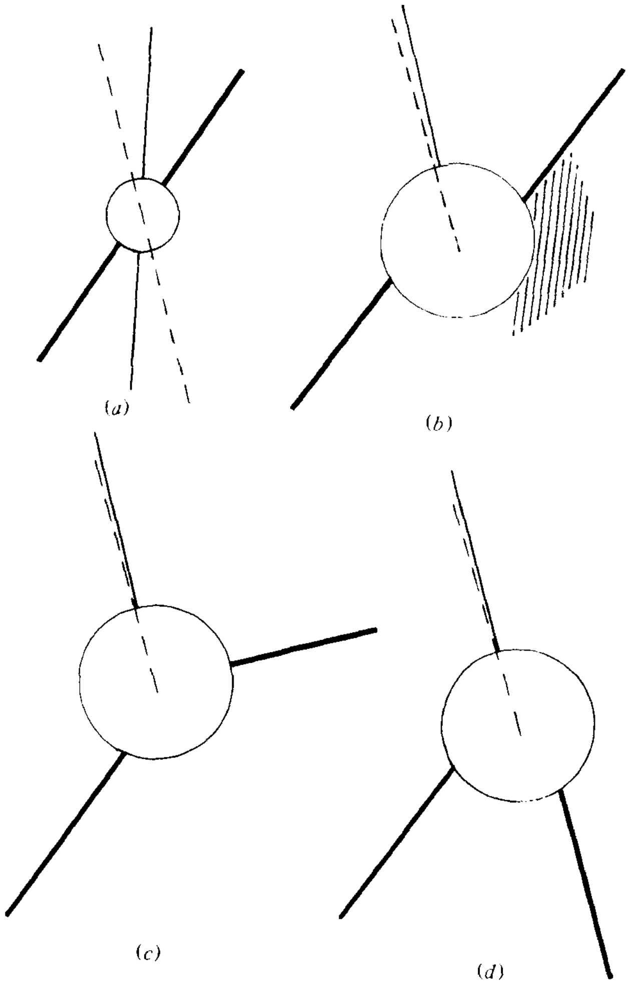


FIGURE 4. A schematic representation of the various types of wave interaction. The solid lines denote the wave fronts (at infinity): ——, large wave; ——, small wave. The broken line indicates the critical angle, and the circle encloses the interaction region. (a) Weak interaction. (b) Strong interaction with $|\mu| > \nu$; the hatched region indicates where the reduced-amplitude small wave exists. (c) Strong interaction with $|\mu| = \nu$; the resonant case. (d) Strong interaction with $|\mu| < \nu$ (for reference only – this solution does not match to a realistic behaviour at infinity).

where $\alpha_i$ are the amplitudes of the solitary waves. For large $\alpha_2/\alpha_1$, this yields

$$\alpha_2 - 2\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} < \tfrac{4}{3}\sin^2\tfrac{1}{2}(\theta_1-\theta_2) < \alpha_2 + 2\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}},$$

approximately. In our criterion we let $\alpha \to 0$ (decreasing the amplitude of the larger wave) and use the results

$$F_1 \sim 1, \quad \lambda_c \sim 1-\tfrac{1}{2}\alpha^2, \quad C_1 \sim \frac{1}{\alpha}, \quad C_2 \sim \tfrac{3}{2}, \quad \nu \sim 1,$$

whence $|\mu| < \nu$ becomes

$$\alpha^2 - 2\alpha(3\epsilon)^{\frac{1}{2}} < 4\sin^2\tfrac{1}{2}(\theta_1-\theta_2) < \alpha^2 + 2\alpha(3\epsilon)^{\frac{1}{2}},$$

and since $\alpha = (3\epsilon_1)^{\frac{1}{2}}$, where $\epsilon_1$ is the amplitude of the larger solitary wave, the criteria are identical.

Thus our theory produces the same sequence of interactions as obtained by Miles (1977*a*). First a weak interaction if $\lambda$ is not close to $\lambda_c$; then a strong interaction if $\lambda - \lambda_c = O(\epsilon^{\frac{1}{2}})$ and $|\mu| > \nu$; finally no realistic solution if $|\mu| < \nu$. These various possibilities are sketched in figure 4.

## 7. Discussion

By using the method of matched asymptotic expansions, the problem of the oblique interaction of a large solitary wave with a small one is examined. As far as this approach allows, we construct the form of the solution both away from and close to the critical angle. The fact that the large solitary wave is not known in closed form makes the study of certain aspects of the problem, e.g. (34)–(36) (or (15)), all the more difficult. Complete solutions to these particular equations would shed further light on the nature of the interaction. However, the part of the interaction that is associated with the distortion of the characteristic of the larger wave is described in some detail. Bearing in mind this limitation, which is circumvented whenever possible by the use of appropriate matching conditions, let us give a brief summary of the results we have obtained.

The basic asymptotic expansion assumes that there exists a representation of the solution as a large wave plus a small one plus interaction terms. The expansion parameter is just the (non-dimensional) amplitude of the smaller wave. It is shown that this expansion can be constructed but that it is not uniquely determined. The arbitrariness is associated with the dominant interaction term, whose form is known but which contains an unknown multiplicative constant (see (17)). This constant is found by matching to the far-field solution, where the dominant term is the small wave together with exponentially small terms associated with the outer reaches of the large wave. The fact that, ultimately, only the asymptotic behaviour of the large wave is required means that the problem remains fairly tractable, although higher-order terms in the basic expansion cannot be obtained. The constant turns out to be proportional to $1/(\lambda - \lambda_c)$, where $\lambda_c = 2F_1/(1+F_1^2+\alpha^2 F_1^4)$ describes the critical angle, and in the special case of two small waves the details agree with those given by Miles (1977*a*). In particular our results generalize the important observation made by Miles that the basic asymptotic expansions are not uniformly valid as $\lambda \to \lambda_c$ ($\lambda_c = 1$ for Miles). For two small waves the non-uniformity occurs as the waves become nearly parallel, but for a large and small wave the angle between them can be as large as about 60°. This raises the question of exactly what physical phenomenon generates the singularity. For $\lambda_c = 1$, many alternatives could be suggested, e.g. relative propagation direction, large region of overlap (and therefore interaction), energy transfer from one wave to the other. Miles (1977*a*) gives no guide, but it should be possible with our more general expression for $\lambda_c$ (involving $F_1$ and $\alpha$) to suggest an interpretation of the critical angle.

In the exponentially small regions of the large wave, the wave profile is a function of $[\alpha(\mathbf{n}_1 . \mathbf{x} - F_1 t)]$ (see (4) and (19)). Thus the group speed can be defined as $(d/d\alpha)(\alpha F_1)$;† with the help of Stokes' result (20) this yields

$$c_{gI} = \tfrac{1}{2}F_1(1+1/F_1^2+\alpha^2 F_1^2),$$

† We use the formal definition more usually associated with $\alpha$ purely imaginary: the same result can be obtained by computing the (local) energy flux and energy density.

where $c_{gI}$ is the group speed of the 'tail' of the large wave. Hence the definition of $\lambda_c$ can be re-cast as

$$c_{gI}\,\mathbf{n}_1.\mathbf{n}_2 = 1; \qquad (67)$$

and noting that the group (and phase) speed of the smaller wave is just unity (to leading order), (67) can be written as

$$(c_{gII}\,\mathbf{n}_2 - c_{gI}\,\mathbf{n}_1).\mathbf{n}_2 = 0, \quad c_{gII} = 1.$$

Geometrically, this states that the propagation of energy, relative to that for the larger wave, is *parallel* to the wave front of the small wave. In other words, on one side of the large wave energy is being propagated directly towards the interaction region; on the other side energy is propagated away. The convention we have adopted implies that energy is entering the interaction region from $\xi > 0$ and it is precisely this energy input that dramatically affects the wave profile. On the other hand, in $\xi < 0$, energy is propagated away along the small wave: this suggests that there might be a reduction in wave amplitude in that region. This is in agreement with the analysis presented here where it is found necessary to construct a solution that requires the small wave to have a reduced amplitude in $\xi < 0$. Thus the essential description of the nonlinear interaction is that of a significant energy transfer directly into and possibly away from this region when $\lambda = \lambda_c$.

The development of the solution for $\lambda = \lambda_c + \epsilon^{\frac{1}{2}}\Lambda$ – the strong interaction case – follows the same procedure as just outlined. The solution requires a basic expansion and a far-field expansion, although there are now important differences between the form of solution in $\xi > 0$ and $\xi < 0$. Close to the interaction (and the large wave) the only term in the asymptotic expansions that can be found is the first. This solution is, not surprisingly, the large wave itself, but it incorporates a phase shift $f(\zeta)$, an arbitrary function of $\zeta$. This unknown function is determined by matching to the far field ($\xi > 0$) where the small wave is given. The exponentially small terms associated with the larger wave are found completely to leading order by imposing appropriate matching and uniform validity conditions. The resulting solution enables $f(\zeta)$ to be expressed in terms of a function $A_0(\zeta)$, provided $A_0(\zeta) \geqslant 0$, where $A_0(\zeta)$ is a hypergeometric function related to $P_\nu^{-\mu}$. The solutions for $\mu \geqslant \nu$, $\mu \leqslant -\nu$ are essentially mirror images of one another in the line $\lambda = \lambda_c$ ($\mu = 0$), and if $|\mu| < \nu$ then we have seen that the resulting solution does not satisfy the conditions that pertain at infinity, and Miles' criterion for the non-existence of a solution is recovered.

The resonant solution given by $\mu = \nu$ (for the case $\mu > 0$), where $A_0 = 0$ at $T = -1$ ($\zeta = -\infty$), is worthy of some further comment. Since $P_\nu^{-\nu}$ is known in closed form (see Erdélyi *et al.* 1953) we can write

$$A_0 = 2^{-\nu}a(1+T)^\nu, \quad T = \tanh \tfrac{1}{2}\sqrt{3}\,\zeta,$$

whence

$$\alpha f(\zeta) = -\nu \ln \tfrac{1}{2}(1 + \tanh \tfrac{1}{2}\sqrt{3}\,\zeta).$$

The peak of the large wave lies along $\xi + f(\zeta) = 0$, to leading order, and as $\zeta \to -\infty$ this becomes

$$\xi \sim \sqrt{3}\frac{\nu}{\alpha}\zeta.$$

Thus the large wave is turned sufficiently so that it is situated between lines $\xi$ = constant, $\zeta$ = constant. This type of interaction is represented in figure 4(*c*), and corresponds to the classical resonant wave (see Miles 1977*a*, *b*).

The problem of determining the solution in $\xi < 0$ when $\lambda - \lambda_c = O(\epsilon^{\frac{1}{2}})$ can be accomplished only indirectly. The difficulty stems from not having available a solution of the set (34)–(36), but we are able to make use of the matching condition involving $f(\zeta)$. In §5 it was argued that, since $f(\zeta)$ is known in terms of $A_0(\zeta)$, the matching to the exponentially small terms is possible only if the leading term is $o(\epsilon)$. This results in a diffusion equation to describe the amplitude in the tail of the large wave. The corresponding solution then matches in all particulars and decays as $\xi \to -\infty$. In consequence, close to the critical angle, the small wave – which is given in the region $\xi > 0$ – does not penetrate through the larger wave to produce a wave of $O(\epsilon)$. The matching condition cited above fails to determine precisely what form this wave takes, but it is tacitly assumed that some information could be gleaned, for example, from a numerical solution of (34)–(36).

Since all other details of the solution appear consistent – and certainly we have produced complete agreement with Miles' work – we can be fairly confident that our interpretation of the form of solution in $\xi < 0$ is correct. After all, considerable headway has been made merely by introducing the exponential tail of the large solitary wave; but by the same token we can expect that some aspects remain unclear.

The solution describing the interaction of the large and small wave, and particularly the bending of the large wave, requires detailed information about the associated Legendre function $P_\nu^{-\mu}$. This limits the usefulness of our results since, for general $\mu$ and $\nu$, $P_\nu^{-\mu}$ is not expressible in a simple closed form. However, one result that is readily available is the phase shift of the large wave. By using the asymptotic behaviour of $P_\nu^{-\mu}(T)$ as $T \to -1$, it is easy to show that the phase shift is

$$-\frac{1}{\alpha}\ln\left[\frac{\Gamma(|\mu|-\nu)\,\Gamma(1+|\mu|+\nu)}{\Gamma(|\mu|)\,\Gamma(1+|\mu|)}\right] \quad (|\mu| > \nu),$$

to leading order in $\epsilon$. This confirms all our previous results, for as $|\mu| \to \infty$ ($\nu$ fixed) the phase shift approaches 0, and as $|\mu| \to \nu$ the phase shift increases without bound.

As a final example of the type of solution derived here (and as a useful synopsis of the details) let us consider the case of $\nu = 1$, for which $P_\nu^{-\mu}$ takes an elementary form. Of course we are interested in $\nu = 1$ for $F_1$ not close to unity† (see figure 3), and this corresponds to $F_1 \simeq 1{\cdot}225$. The critical angle is about 58° (see figure 1), and so for an oblique interaction not close to this angle the solution (for $\eta$, say) is

$$\eta \sim \eta_{\rm I}(\xi) - \epsilon^{\frac{1}{2}}\lambda_c \frac{1+2\lambda F_1 + \alpha^2 F_1^4}{2F_1(\lambda-\lambda_c)}\phi_{\rm II0}(\zeta)\,\eta_{{\rm I}\xi} + \epsilon(\eta_{\rm II0}(\zeta) + \hat{\eta}_2),$$

which is actually valid for all $\nu$. The important point about this solution is that the dominant interaction term is $O(\epsilon^{\frac{1}{2}})$, whereas the wave producing it is $O(\epsilon)$. Close to the critical angle we have $\lambda = \lambda_c + \epsilon^{\frac{1}{2}}\Lambda$, with

$$\eta \sim \eta_{\rm I}(\xi + f(\zeta)), \quad \phi \sim \phi_{\rm I}(\xi + f(\zeta), z),$$

where

$$\alpha f(\zeta) = \ln\left[(1+|\mu|)/(T+|\mu|)\right], \quad T = \tanh \tfrac{1}{2}\sqrt{3}\,\zeta,$$

† Note that, even though $\nu = 1$, we maintain $g = o(1)$ (under our assumption: see §5) and hence the small wave is essentially absent in $\xi < 0$ for a strong interaction.

and $\mu = \sqrt{\frac{1}{3}}\, C_1 \Lambda$, $\alpha \simeq 0{\cdot}97$, $C_1 \simeq 1{\cdot}6$. If $|\mu| > 1$, with $f(\infty) = 0$, then as $\zeta \to -\infty$ the phase shift is

$$-\frac{1}{\alpha} \ln [(|\mu| + 1)/(|\mu| - 1)];$$

if $|\mu| = 1$ we have the resonant wave with an infinite phase shift, and the wave lies along $\xi \sim 1{\cdot}79\zeta$, $\zeta \to -\infty$.

In conclusion, this work has shown how the interaction of the two waves has an associated critical angle which can be interpreted in terms of energy transport. The strong interaction can produce significant, that is $O(1)$, changes to the large wave. It would seem to be quite possible to measure the angle between the waves, when the small wave distorts the larger, and thereby obtain values for the critical angle. Of course, one can envisage difficulties that arise out of the unsteady character of the experimental (or natural) set-up, and therefore it might be necessary to think in terms of a periodic phenomenon. In this case we might hope that our theory gives the essential description of the interaction in a local sense, if the waves are fairly well spaced. To some extent it should also be possible to estimate the phase shift of the larger wave, and to seek for the special resonant interaction. Some of these points are currently under examination in the Department of Ocean Engineering at Newcastle.

Finally, a direct application of the theory is in the prediction of wave breaking. By using either the exact condition for the onset of breaking or an appropriate amplitude-to-depth ratio, the effect of wave interaction on wave breaking can be examined. Even apart from the strong-interaction case, we can anticipate a surprisingly large effect, since the dominant interaction term is $O(\epsilon^{\frac{1}{2}})$ when the small wave is only $O(\epsilon)$. Further, in this case, the interaction term is known completely, although it does only shift the large wave; this, however, could be quite significant if the local depth is changing. In both the strong and weak interactions, further information about the interaction process is not available. This would appear to be the next and main avenue of study, presumably by the numerical solution of the relevant interaction equations or possibly the full problem itself. Certainly the oblique interaction of two arbitrary-amplitude solitary waves is currently outside the scope of analytical treatment, and only a numerical approach gives any hope of success.

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